A new counting methods, including the issue of counting labelled self-complementary graphs

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abstract

Harary and Palmer announced an enumeration problem of labelled self-complementary graphs at the end of their book (Graphical Enumeration, Academic Press, New York and London, 1973). This paper resolves this problem. A method for solving this problem leads to the derivation of following formulas: (a) A formula on the number of labelled graphs with the given order of automorphism groups of those graphs. (b) A formula on the number of unlabelled graphs with the given order of automorphism groups of those graphs. (c) A formula on the number of labelled self-complementary graphs with the given order of automorphism groups of those graphs. (d) A formula on the number of unlabelled self-complementary graphs with the given order of automorphism groups of those graphs.

Keywords: Graphical enumeration, Graph, Self-complementary graph, Automorphism group, Generating function

1 Introduction

This paper considers finite graphs. They do not have multiple edges nor loops on vertices. Harary and Palmer[5] announced many graphical enumeration problems at the end of their book, including an enumeration problem of labelled self-complementary graphs. This paper resolves the latter problem. A method for solving this problem leads to the derivation of following formulas: (a) A formula on the number of labelled graphs with the given order of automorphism groups of those graphs. (b) A formula on the number of unlabelled graphs with the given order of automorphism groups of those graphs. (c) A formula on the number of labelled self-complementary graphs with the given order of automorphism groups of those graphs. (d) A formula on the number of unlabelled self-complementary graphs with the given order of automorphism groups of those graphs.

A formula on the number of unlabelled graphs was given by Harary[3] by using Pólya theorem[6]. This number is found by taking the sum of the numbers in (b) over the orders of the automorphism groups of those graphs. The enumeration of unlabelled self-complementary graphs dates back to the Read's investigation[7] in 1963 and Read[7] gave a formula on the number of unlabelled self-complementary graphs, by using De Bruijn's theorem[1,2]. This number is found by taking the sum of the numbers in (d) over the orders of the automorphism groups of those self-complementary graphs.

In the section 2 we consider to assign a number to each labelled graph. This assignment is very useful in the sense that from the number having been assigned to a graph, the graph can be uniquely determined. The section 3 discusses about the auotmorphism groups of graphs. The section 4 treats the

enumerations of labelled graphs and unlabelled graphs with the orders of automorphism groups of those graphs. In the section 6 we treat the enumerations of labelled self-complementary graphs, which is a main purpose in this paper, and the unlabelled self-complementary graphs.

2 The indexes of graphs

For a positive integer n let V be the set $\{1, 2, ..., n\}$ and let $\binom{V}{2} = \{\{i, j\} \subset V | i \neq j\}$. We here denote each element $\{i, j\}$ of $\binom{V}{2}$ by ij when $i < j$ and by ji when $i > j$. For a subset E of $\binom{V}{2}$ we call the pair (V, E) a labelled graph or a graph on V. $|V|$ is called the order of the graph and E is called the edge-set of the graph. We define a bijection w from $\binom{V}{2}$ to $\{2^0 = 1, 2^1, 2^2, \cdots, 2^{\lambda-1}\}$ by

$$w(ij) = 2^{\lambda - p_{ij}} \qquad ij \in \binom{V}{2}, \tag{1}$$

where $\lambda = \binom{n}{2}$ and $p_{ij} = \frac{(2n-i)(i-1)}{2} + (j - i)$ for $ij \in \binom{V}{2}$. Let $\mathcal{G}_n$ be the set of all labelled graphs of order n and let $\mathfrak{N}_n = \{0, 1, 2, 3, \cdots, 2^\lambda - 1\}$. We define a map N from $\mathcal{G}_n$ to $\mathfrak{N}_n$ by

$$N(G) = \begin{cases} \displaystyle\sum_{ij \in E} w(ij) = \sum_{ij \in E} 2^{\lambda - p_{ij}} & \text{if } E \text{ is not empty} \\ 0 & \text{if } E \text{ is empty.} \end{cases} \tag{2}$$

$N(G)$ is called the index of G. We consider the binary number representation of an integer L satisfying $0 \leqq L \leqq 2^\lambda - 1$ and the binary number of L is denoted by $L^{(2)}$, where $L^{(2)}$ is a number of $\binom{n}{2}$ figures. For example a decimal number 29 is represented as the binary number 011101, whose leading digit "0" is not omitted. In the binary number $L^{(2)}$ of L, the p_{ij}-th digit is written by $L^{(2)}_{ij}$ for each element $ij \in \binom{V}{2}$. The binary number of the index $N(G)$ of a labelled graph G is denoted by $N^{(2)}(G)$ and its p_{ij}-th digit is written by $N^{(2)}_{ij}(G)$. The index $N(G)$ of a labelled graph $G = (V, E)$ has the following property:

$$N^{(2)}_{ij}(G) = 1 \quad \text{if and only if} \quad ij \in E. \tag{3}$$

We first give the following result.

Theorem 1. *The map N is a bijection.*

Proof. Consider $L \in \mathfrak{N}_n$. If $L = 0$, then for the empty graph G whose edge-set is empty, we have $N(G) = L$. Suppose $L \neq 0$. Then the graph G on V having the set $E = \{ij \in \binom{V}{2} | L^{(2)}_{ij} = 1\}$ as its edge-set is an element of $\mathcal{G}_n$ and it is obvious that $N(G) = L$ holds. For $G_1, G_2 \in \mathcal{G}_n$, if G_1 and G_2 are different, then $N^{(2)}(G_1)$ is not equal to $N^{(2)}(G_2)$. This fact implies that $N(G_1)$ is not equal to $N(G_2)$. Hence N is bijection. □

As an illustration, consider a graph $G = (V, E = \{13, 14, 23, 34\})$ on $V = \{1, 2, 3, 4\}$. Since $p_{13} = 2, p_{14} = 3, p_{23} = 4$ and $p_{34} = 6$, it is easy to check that $N(G) = 29$ and $N^{(2)}(G) = 011101$. Conversely, a binary number $L^{(2)} = 011101$ yields the labelled graph with the index L, since $L^{(2)}_{13} = L^{(2)}_{14} = L^{(2)}_{23} = L^{(2)}_{34} = 1$

and $L_{12}^{(2)} = L_{24}^{(2)} = 0$. This graph is what was given in the beginning of this illustration.

The complement $\overline{G}$ of a graph $G = (V, E)$ is the graph on V with edge-set $\binom{V}{2} - E$. The index $N(\overline{G})$ of $\overline{G}$ is $2^\lambda - 1 - N(G)$. It is easy to see that $N^{(2)}(\overline{G})$ is obtained by interchanging 0 and 1 in $N^{(2)}(G)$.

3 Graph and group

Let A be the symmetric group on V. In this paper we assume $n \geqq 3$. Each permutation α in A induces a permutation α' which acts on $\binom{V}{2}$ such that for every element $ij \in \binom{V}{2}$,

$$\alpha'\{i, j\} = \{\alpha i, \alpha j\}.$$

The permutation group $A' = \{\alpha' | \alpha \in A\}$ is called the pair group of A. The degree of A' is λ and $A \cong A'$, since $n \geqq 3$. For a graph $G = (V, E)$ and $\alpha \in A$, the graph which has $\{\{\alpha i, \alpha j\} \in \binom{V}{2} | ij \in E\}$ as its edge-set is denoted by αG. For a graph G, an element α in A satisfying $\alpha G = G$ is called the automorphism of G. The automorphism group $\Gamma(G) = \{\alpha \in A | \alpha G = G\}$ of G is called the group of G, and the number of elements which belong to the group is called the group-order.

A graph G is said to be self-complementary if G and $\overline{G}$ are isomorphic, that is, if there exists $\alpha \in A$ such that $\alpha G = \overline{G}$ holds. It is well-known that every self-complementary graph G has $n \equiv 0, 1 (\mod 4)$ vertices. Note that while a self-complementary graph G is isomorphic to $\overline{G}$, G and $\overline{G}$ do not have the same index. Let $\mathcal{S}_n$ be the set of all labelled self-complementary graphs on V. Then we define a set $A_S = \{\alpha \in A | \exists G \in \mathcal{S}_n, \alpha G = \overline{G}\}$ and we also define a set $A_S(G) = \{\alpha \in A_S | \alpha G = \overline{G}\}$ for $G \in \mathcal{S}_n$. For $\alpha' \in A'$ let $Z(\alpha')$ be the set of all cycles in the disjoint cycle decomposition of α'. Then the following lemma can easily be seen from the definitions of $\Gamma(G)$ and $A_S(G)$.

Lemma 1. @

(i) *For $\alpha \in A$ and $G = (V, E) \in \mathcal{G}_n$, α is an element of $\Gamma(G)$ if and only if for each $z \in Z(\alpha')$,*

$$ij \in E \quad \text{if and only if} \quad z(ij) \in E, \quad \forall ij \in z \tag{4}$$

holds.

(ii) *For $\alpha \in A$ and $G = (V, E) \in \mathcal{G}_n$, G is a self-complementary graph satisfying $\alpha G = \overline{G}$, that is, $\alpha \in A_S(G)$ if and only if for each $z \in Z(\alpha')$,*

$$ij \in E \quad \text{if and only if} \quad z(ij) \notin E, \quad \forall ij \in z \tag{5}$$

holds.

Lemma 2. @

(i) *For $\alpha \in A$ and $G \in \mathcal{G}_n$, α is an element of $\Gamma(G)$ if and only if for each $z \in Z(\alpha')$,*

$$N_{ij}^{(2)}(G) = 1 \text{ for } \forall ij \in z \quad \text{or} \quad N_{ij}^{(2)}(G) = 0 \text{ for } \forall ij \in z \tag{6}$$

holds.

(ii) *For $\alpha \in A$ and $G \in \mathcal{G}_n$, G is a self-complementary graph satisfying $\alpha G = \overline{G}$, that is, $\alpha \in A_S(G)$ if and only if for each $z \in Z(\alpha')$,*

$$N_{ij}^{(2)}(G) + N_{z(ij)}^{(2)}(G) = 1 \quad \text{for } \forall ij \in z \tag{7}$$

holds.

Proof. (i) is immediate from the combination of (3) and (i) in Lemma 1. We next prove (ii). Let E be the edge-set of G. Then (3) can be written as follows:

$$z(ij) \notin E \quad \text{if and only if} \quad N_{z(ij)}^{(2)}(G) = 0. \tag{8}$$

Therefore, it is easy to see from (3), (8) and (ii) in Lemma 1 that the desired result is obtained. □

Theorem 2. *Let $\mathcal{K}(P)$ be the set of all labelled graphs of order n all of which have a property P and let ξ be a positive integer. If the group-order of the group of every graph in $\mathcal{K}(P)$ is ξ, then the number of graphs which belong to $\mathcal{K}(P)$ and which are not mutually isomorphic is ,that is, the number of unlabelled graphs in $\mathcal{K}(P)$ is given by*

$$\frac{\xi|\mathcal{K}(P)|}{n!}. \tag{9}$$

Proof. Let $l(G)$ be the number of ways of labelling a graph G. Then using a well-known $l(G) = \frac{n!}{|\Gamma(G)|}$ [5;p4], we get the equality (9), since $|\Gamma(G)| = \xi$ for $G \in \mathcal{K}(P)$. This complete the proof. □

We have the following results with respect to self-complementary graphs.

Theorem 3. *Suppose that G is an element of $\mathcal{S}_n$. Then for any $\alpha \in A_S(G)$, $\alpha^{-1}A_S(G) = \Gamma(G)$ holds.*

Proof. It follows that for any $\beta \in A_S(G)$, $(\alpha^{-1}\beta)G = \alpha^{-1}(\beta G) = \alpha^{-1}\overline{G} = G$ holds. Therefore we have $\alpha^{-1}\beta \in \Gamma(G)$, which implies $\alpha^{-1}A_S(G) \subseteq \Gamma(G)$. Conversely, Any element γ of $\Gamma(G)$ can be written as $\gamma = \alpha^{-1}\alpha\gamma$. Thus $(\alpha\gamma)G = \alpha(\gamma G) = \alpha G = \overline{G}$ is obtained. Therefore, we have $\alpha\gamma \in A_S(G)$, that is, $\gamma \in \alpha^{-1}A_S(G)$, which implies $\Gamma(G) \subseteq \alpha^{-1}A_S(G)$. Hence $\alpha^{-1}A_S(G) = \Gamma(G)$ holds. □

Theorem3 gives the following.

Corollary 1. *For $G \in \mathcal{S}_n$, $|A_S(G)| = |\Gamma(G)|$ holds.*

For $\alpha \in A$ let $m_k(\alpha)$ be the number of cycles of length k in the disjoint cycle decomposition of α. Then we prove

Theorem 4. *Let n be a positive integer satisfying $n \equiv 0, 1 (\bmod\ 4)$. Then α is an element of A_S if and only if the following condition holds:*

$$m_1(\alpha) \qquad \leq 1 \quad \textit{and} \tag{10}$$

$$m_k(\alpha) \qquad \begin{cases} \geq 1 & \textit{if } k \equiv 0 \pmod 4 \\ = 0 & \textit{if } k \not\equiv 0 \pmod 4 \end{cases} \tag{11}$$

for $k = 2, 3, ..., n$.

Proof. Suppose $\alpha \in A_S$. Then there exists $G \in \mathcal{S}_n$ satisfying $\alpha G = \overline{G}$. Accordingly, it is easy to see that the length of each cycle in $Z(\alpha')$ must be even. This fact implies $m_1(\alpha) \leq 1$ and $m_k(\alpha) = 0$ for odd $k(\geqq 3)$. If α has a cycle of length $2k$ providing k is odd, then it generates a cycle of odd length k in α' which contradicts the fact that the length of each cycle in $Z(\alpha')$ is even.

On the other hand, suppose that the permutation α of A satisfies the conditions (10) and (11). Then the length of each cycle in $Z(\alpha')$ for the permutation $\alpha' \in A'$ induced by α is always even. Accordingly, we can construct the binary number $L^{(2)}$ of $\binom{n}{2}$ figures satisfying

$$L^{(2)}_{ij} + L^{(2)}_{z(ij)} = 1, \quad \forall ij \in z$$

for $z \in Z(\alpha')$. If we consider the labelled graph G on V which has $E = \{ij \in \binom{V}{2} | L^{(2)}_{ij} = 1\}$ as its edge-set, (ii) in Lemma 2 tells us that G is a self-complementary graph of order n satisfying $\alpha G = \overline{G}$. Hence we have $\alpha \in A_S$. $\square$

4 Counting of graphs

For each $z \in Z(\alpha')$ in $\alpha \in A$, we consider

$$W(z) = \sum_{ij \in z} w(ij). \tag{12}$$

Then to $\alpha \in A$ there corresponds the following polynomial of x:

$$F^{(n)}_{\alpha}(x) = \prod_{z \in Z(\alpha')} f^{(n)}_z(x), \tag{13}$$

where

$$f^{(n)}_z(x) = 1 + x^{W(z)}. \tag{14}$$

Let us consider the generating function for labelled graphs of order n

$$F^{(n)}(x) = \sum_{\alpha \in A} F^{(n)}_{\alpha}(x). \tag{15}$$

Let $c^{(\alpha)}_L$ be the coefficient of x^L in the power series expansion of $F^{(n)}_{\alpha}(x)$ for $\alpha \in A$, and let c_L be the coefficient of x^L in the power series expansion of $F^{(n)}(x)$. Then the following equality is an immediate consequence of (15):

$$c_L = \sum_{\alpha \in A} c^{(\alpha)}_L. \tag{16}$$

Let $f(x)$ be a polynomial of x. Then by the symbol $a_n x^n \vDash f(x)$, $a_n x^n$ will mean the term in the polynomial $f(x)$. On the other hand, If $a_n x^n$ is not any term of $f(x)$, that is, if $a_n x^n \nvDash f(x)$, we use the convention $a_n = 0$.

Let $lg(n) = |\mathcal{G}_n|$. Then $lg(n)$ is equal to 2^{λ}, since $|\mathcal{G}_n| = 2^{\lambda}$, as well-known. Consider an $n!$ by $lg(n)$ matrix $M(\mathcal{G}_n) = ||c^{(\alpha)}_L||$, which has a row for each $\alpha \in A$ and a column for each $G \in \mathcal{G}_n$, provided that $c^{(\alpha)}_L = 0$ when $c^{(\alpha)}_L x^L \nvDash F^{(n)}_{\alpha}(x)$. Since by Theorem 1 there is a one-to-one correspondence between $\mathcal{G}_n$ and $\mathfrak{N}_n$, the column corresponding to $G \in \mathcal{G}_n$ is referred to as the $N(G)$-column.

Property 1. *In the matrix $M(\mathcal{G}_n)$, if $c_L^{(\alpha)} \neq 0$, then $c_L^{(\alpha)} = 1$ holds.*

Proof. Consider $\alpha \in A$. If $L = 0$, then since the constant term in the power series expansion of $F_\alpha^{(n)}(x)$ is 1, we have $c_0^{(\alpha)} = 1$. Suppose that $L \neq 0$. Then note first that $c_L^{(\alpha)} x^L \vDash F_\alpha^{(n)}(x)$, since $c_L^{(\alpha)} \neq 0$. From the observation of (13) there exists a subset $K = \{z_1, z_2, \cdots, z_t\}$ of $Z(\alpha')$ such that $L = W(z_1) + W(z_2) + \cdots + W(z_t)$. It can easily seen that the binary number $L^{(2)}$ satisfies

$$L_{ij}^{(2)} = 1,\ \forall ij \in z, \forall z \in K$$
$$L_{ij}^{(2)} = 0,\ \forall ij \in z, \forall z \in Z(\alpha') - K.$$

Thus if we consider the labelled graph $G \in \mathcal{G}_n$ whose edge-set is $\{ij \in \binom{V}{2} | L_{ij}^{(2)} = 1\}$, then it follows that its index $N(G)$ is L and that $\alpha G = G$ holds by (i) in Lemma 2. Since $L^{(2)}$ is determined by L, vice versa, L is determined by $L^{(2)}$, the subset K being discussed just above can be uniquely determined by L. This implies that $c_L^{(\alpha)} = 1$. $\square$

This property states that for $\alpha \in A$, every coefficient of the power series expansion of $F_\alpha^{(n)}(x)$ is 1.

Property 2. *The coefficient c_L of x^L in $F^{(n)}(x)$ is equal to the group-order of the group of labelled graph having the index L.*

Proof. In the matrix $M(\mathcal{G}_n)$, (16) denotes the L-th column sum. The L-th column is associated with the labelled graph G having the index $N(G) = L$, and (16) becomes $c_L = \sum_{\alpha \in \Gamma(G)} c_L^{(\alpha)}$. Applying Property 1, we get $c_L = |\Gamma(G)|$. $\square$

If we consider two sets: $\mathcal{O}(\mathcal{G}_n) = \{|\Gamma(G)|| G \in \mathcal{G}_n\}$ and $\mathcal{C}(\mathcal{G}_n) = \{c_L | c_L x^L \vDash F^{(n)}\}$, then Property 2 gives the following theorem, which states that the list of the group-orders of the groups of all graphs of order n can be obtained by expanding $F^{(n)}(x)$.

Theorem 5. $\mathcal{O}(\mathcal{G}_n) = \mathcal{C}(\mathcal{G}_n)$ *holds.*

For $\xi \in \mathcal{O}(\mathcal{G}_n)$, let $lg(n;\xi)$ be the number of *labelled* graphs G of order n such that $|\Gamma(G)| = \xi$, and let $g_\xi^{(n)}$ be the number of *unlabelled* graphs G of order n such that $|\Gamma(G)| = \xi$. For a positive integer ξ, put $\mathcal{L}_\xi(\mathcal{G}_n) = \{L | c_L = \xi,\ c_L x^L \vDash F^{(n)}(x)\}$. Then we have the following three theorems.

Theorem 6. *The equality*

$$lg(n;\xi) = |\mathcal{L}_\xi(\mathcal{G}_n)|, \quad \xi \in \mathcal{C}(\mathcal{G}_n) \tag{17}$$

holds.

Proof. As seen from the statement of Property 2 and its proof, each labelled graph G in $\mathcal{G}_n$ having $|\Gamma(G)| = \xi$ is associated with the $N(G)$-th column of $M(\mathcal{G}_n)$ whose column sum is ξ, and each element of $\mathcal{L}_\xi(\mathcal{G}_n)$ is also associated with a column of $M(\mathcal{G}_n)$ whose column sum is ξ. Accordingly, $lg(n;\xi)$ is the number of columns of $M(\mathcal{G}_n)$ whose column sum is ξ and also $|\mathcal{L}_\xi(\mathcal{G}_n)|$ is the number of columns of $M(\mathcal{G}_n)$ whose column sum is ξ. Hence we obtain the equality (17), noting Theorem 5. $\square$

Theorem 7. *@ The equality*

$$g_{\xi}^{(n)} = \frac{\xi}{n!}|\mathcal{L}_{\xi}(\mathcal{G}_n)|, \qquad \xi \in \mathcal{C}(\mathcal{G}_n) \tag{18}$$

holds.

Proof. In Theorem 2, let P denote the property that the group-order of the group of a labelled graph in $\mathcal{G}_n$ is ξ. Then $\mathcal{K}(P)$ is the set of all labelled graphs G in $\mathcal{G}_n$ satisfying $|\Gamma(G)| = \xi$, that is, $|\mathcal{K}(P)| = lg(n;\xi)$. Hence using Theorem 2, Theorem 5 and Theorem 6, the equality (18) is obtained. □

Let $g(n)$ be the number of unlabelled graphs of order n. A formula on $g(n)$ has been given by Harary[3]. But an enumeration method having been here presented will derive a formula on $g(n)$, which is stated in the following.

Theorem 8. *The number $g(n)$ of unlabelled graphs of order n is given by*

$$g(n) = \sum_{\xi \in \mathcal{C}(\mathcal{G}_n)} \frac{\xi}{n!}|\mathcal{L}_{\xi}(\mathcal{G}_n)|. \tag{19}$$

Proof. Since $g(n) = \sum_{\xi \in \mathcal{O}(\mathcal{G}_n)} g_{\xi}^{(n)}$, using Theorem 5 and Theorem 7, we get the equality (19). □

5 Numerical results for graphs

We have the numerical examples for $n \geqq 3$, since $n \geqq 3$ in this paper. Table 1 gives the numerical examples for labelled graphs. Each pair $(\xi, lg(n;\xi))$ in the Table 1 shows the number $lg(n;\xi)$ of labelled graphs of order n with the group-order ξ for a few small values of $n \geqq 3$. The last column of Table 1 shows the number $lg(n)$ of labelled graphs of order n, which is well-known. On the other hand, Table 2 gives the numerical examples for unlabelled graphs. Each pair $(\xi, g_{\xi}^{(n)})$ in the Table 2 shows the number $g_{\xi}^{(n)}$ of unlabelled graphs of order n with the group-order ξ. The last column of Table 2 shows the number $g(n)$ of unlabelled graphs of order n.

Table 1: Labelled graphs

n	$(\xi,\ lg(n;\xi))$	$lg(n)$
3	$(2,6),(6,2)$	2^3
4	$(2,36),(4,12),(6,8),(8,6),(24,2)$	2^6
5	(2,660),(4,180),(6,40),(8,60),(10,12), (12,60), (24,10),(120,2)	2^{10}
6	(1,5760), (2,16560), (4,6480), (6,960), (8,1260), (10,144), (12,1080), (16,270), (24,60), (36,40), (48,120), (72,20), (120,12), (720,2)	2^{15}
7	(1,766080), (2,892080), (4,312480), (6,31920), (8,46620), (10,1008), (12,29400), (14,720), (16,6300), (20,1008), (24,5040), (36,840), (48,2940), (72,280), (120,84), (144,210), (240,126), (720,14), (5040,2)	2^{21}

6 Counting of self-complementary graphs

In this section we make a consideration about the enumeration of self-complementary graphs. This enumeration is made along the argument in the preceding section. Let n, throughout this section, be an integer satisfying $n \equiv 0, 1 (\mod 4)$ and $n \geqq 4$. For $\alpha \in A_S$, the length of each cycle in $Z(\alpha')$ is always even. We denote a cycle $z \in Z(\alpha')$ by $z = (i_1j_1, i_2j_2, \cdots, i_{k-1}j_{k-1}, i_kj_k)$. The cycle can be represented in some forms in the usual cyclic representation, as well-known. For example, the cycle$(12, 23, 34, 14)$ can also be written $(23, 34, 14, 12)$, $(34, 14, 12, 23)$ or $(14, 12, 23, 34)$. We assume that the leading position of z is occupied by the element at which the value of w is the maximum among those elements of z. Let $P_1(z) = \{i_1j_1, i_3j_3, \cdots, i_{k-1}j_{k-1}\}$ be the set of elements in the odd position in the cycle z and let $P_2(z) = \{i_2j_2, i_4j_4, \cdots, i_kj_k\}$ be the set of elements in the even position in z. Then, note that for $ij \in z$, $ij \in P_1(z)$ if and only if $z(ij) \in P_2(z)$. For each $z \in Z(\alpha')$ in $\alpha \in A_S$, we consider

$$W_1(z) = \sum_{ij \in P_1(z)} w(ij) \quad \text{and} \quad W_2(z) = \sum_{ij \in P_2(z)} w(ij). \tag{20}$$

Then to $\alpha \in A_S$ there corresponds the following polynomial of x:

$$U_\alpha^{(n)}(x) = \prod_{z \in Z(\alpha')} u_z^{(n)}(x), \tag{21}$$

where

$$u_z^{(n)}(x) = x^{W_1(z)} + x^{W_2(z)}, \quad z \in Z(\alpha'). \tag{22}$$

Table 2: Unlabelled graphs

n	$(\xi,\ g_{\xi}^{(n)})$	$g(n)$
3	$(2,2),(6,2)$	4
4	$(2,3),(4,2),(6,2),(8,2),(24,2)$	11
5	$(2,11),(4,6),(6,2),(8,4),(10,1),(12,6),(24,2),(120,2)$	34
6	(1,8), (2,46), (4,36), (6,8), (8,14), (10,2), (12,18), (16,6), (24,2), (36,2), (48,8), (72,2), (120,2), (720,2)	156
7	(1,152), (2,354), (4,248), (6,38), (8,74), (10,2), (12,70), (14,2), (16,20), (20,4), (24,24), (36,6), (48,28), (72,4), (120,2), (144,6), (240,6), (720,2), (5040,2)	1044

Let us consider the generating function for labelled self-complementary graphs of order n

$$U^{(n)}(x) = \sum_{\alpha \in A_S} U_{\alpha}^{(n)}(x). \tag{23}$$

Let $lsc(n) = |\mathcal{S}_n|$ be the number of labelled self-complementary graphs of order n. Then we shall show that $lsc(n)$ is equal to the number of terms in the power series expansion of $U^{(n)}(x)$. For $G \in \mathcal{S}_n$, there exists $\alpha \in A_S$ satisfying $\alpha G = \overline{G}$. It is seen from (ii) in Lemma 2 that the index $N(G)$ of G satisfies

$$N_{ij}^{(2)}(G) + N_{z(ij)}^{(2)}(G) = 1, \quad \forall ij \in z \tag{24}$$

for each $z \in Z(\alpha')$. For $z \in Z(\alpha')$, if we put $P(z) = \{ij \in z | N_{ij}^{(2)}(G) = 1\}$, then it follows from (24) that $P(z)$ agrees with either $P_1(z)$ or $P_2(z)$. Furthermore, if we put $W(z) = \sum_{ij \in P(z)} w(ij)$ for each $z \in Z(\alpha')$, then it is easy to check that $N(G) = \sum_{z \in Z(\alpha')} W(z)$. Thus $\prod_{z \in Z(\alpha')} x^{W(z)} = x^{N(G)}$ is a term in the power series expansion of $U_{\alpha}^{(n)}(x)$, which implies that this term is appeared in the power series expansion of $U^{(n)}$. Since N is a bijection from $\mathcal{G}_n$ to $\mathfrak{N}_n$ by Theorem 1, we have the following theorem.

Theorem 9. *Let $\mathcal{L}(\mathcal{S}_n) = \{L | d_L x^L \vDash U^{(n)}(x)\}$. Then the equality $lsc(n) = |\mathcal{L}(\mathcal{S}_n)|$ holds.*

Let $d_L^{(\alpha)}$ be the coefficient of x^L in the power series expansion of $U_{\alpha}^{(n)}(x)$ for $\alpha \in A_S$, and let d_L be the coefficient of x^L in the power series expansion of

$U^{(n)}(x)$. Then the following equality is an immediate consequence of (23):

$$d_L = \sum_{\alpha \in A_S} d_L^{(\alpha)}. \tag{25}$$

Consider an $|A_S|$ by $lsc(n)$ matrix $M(\mathcal{S}_n) = ||d_L^{(\alpha)}||$, which has a row for each $\alpha \in A_S$ and a column for each $G \in \mathcal{S}_n$, provided that $d_L^{(\alpha)} = 0$ when $d_L^{(\alpha)} x^L \nvDash U_\alpha^{(n)}(x)$. $M(\mathcal{S}_n)$ may be regarded as a submatrix of $M(\mathcal{G}_n)$, if $c_L^{(\alpha)}$ in the latter is replaced by $d_L^{(\alpha)}$ in the former. Thus the column corresponding to $G \in \mathcal{S}_n$ is referred to as the $N(G)$-column.

Property 3. *In the matrix $M(\mathcal{S}_n)$, if $d_L^{(\alpha)} \neq 0$, then $d_L^{(\alpha)} = 1$ holds.*

Proof. Consider $\alpha \in A_S$. Note first that $d_L^{(\alpha)} x^L \vDash U_\alpha^{(n)}(x)$, since $d_L^{(\alpha)} \neq 0$. Also, note that $L \neq 0$, since we assume that $n \geqq 4$. Put $Z(\alpha') = \{z_1, z_2, \cdots, z_k\}$ and put $I = \{1, 2\}$. Then from the observation of (21) there exists a $k-$tuple $(l_1, l_2, \cdots, l_k) \in \underbrace{I \times I \times \cdots \times I}_{k}$ such that $L = W_{l_1}(z_1) + W_{l_2}(z_2) + \cdots + W_{l_k}(z_k)$. It can easily seen that the binary number $L^{(2)}$ satisfies

$$L_{ij}^{(2)} = 1, \ \forall ij \in P_{l_1}(z_1) \cup P_{l_2}(z_2) \cup \cdots \cup P_{l_k}(z_k)$$
$$L_{ij}^{(2)} = 0, \ \forall ij \in P_{l_1'}(z_1) \cup P_{l_2'}(z_2) \cup \cdots \cup P_{l_k'}(z_k),$$

where $l_h' \neq l_h, \ l_h' \in I \, (h = 1, 2, \cdots, k)$. Thus if we consider the labelled graph $G \in \mathcal{G}_n$ whose edge-set is $\{ij \in \binom{V}{2} | L_{ij}^{(2)} = 1\}$, then it is obvious that its index $N(G)$ is L, and (ii) in Lemma 2 tells us that G is a self-complementary graph satisfying $\alpha \in A_S(G)$. As seen at the end of the proof of Property 1, since $L^{(2)}$ is determined by L, vice versa, L is determined by $L^{(2)}$, the k-tuple $(l_1, l_2, \cdots, l_k)$ being discussed just above can be uniquely determined by L. This implies that $d_L^{(\alpha)} = 1$. □

This property states that for $\alpha \in A_S$, every coefficient of the power series expansion of $U_\alpha^{(n)}(x)$ is 1.

Property 4. *The coefficient d_L of x^L in $U^{(n)}(x)$ is equal to the group-order of the group of labelled self-complementary graph having the index L.*

Proof. In the matrix $M(\mathcal{S}_n)$, (25) denotes the L-th column sum. The L-th column is associated with the labelled self-complementary graph G having the index $N(G) = L$, and (25) becomes $d_L = \sum_{\alpha \in A_S(G)} d_L^{(\alpha)}$. Applying Property 3, we get $d_L = |A_S(G)|$. Hence we have $d_L = |\Gamma(G)|$ from Corollary 1. □

If we consider two sets: $\mathcal{O}(\mathcal{S}_n) = \{|\Gamma(G)| | G \in \mathcal{S}_n\}$ and $\mathcal{D}(\mathcal{S}_n) = \{d_L | d_L x^L \vDash U^{(n)}\}$, then Property 4 gives the following theorem, which states that the list of the group-orders of the groups of all self-complementary graphs of order n can be obtained by expanding $U^{(n)}(x)$.

Theorem 10. $\mathcal{O}(\mathcal{S}_n) = \mathcal{D}(\mathcal{S}_n)$ *holds.*

For $\xi \in \mathcal{O}(\mathcal{S}_n)$, let $lsc(n;\xi)$ be the number of *labelled* self-complementary graphs G of order n such that $|\Gamma(G)| = \xi$, and let $s_\xi^{(n)}$ be the number of *unlabelled* self-complementary graphs G of order n such that $|\Gamma(G)| = \xi$. For a positive integer ξ, put $\mathcal{L}_\xi(\mathcal{S}_n) = \{L | d_L = \xi,\ d_L x^L \vDash U^{(n)}(x)\}$. Then we have the following three theorems.

Theorem 11. *The equality*

$$lsc(n;\xi) = |\mathcal{L}_\xi(\mathcal{S}_n)|, \quad \xi \in \mathcal{D}(\mathcal{S}_n) \tag{26}$$

holds.

Proof. As seen from the statement of Property 4 and its proof, each labelled graph G in $\mathcal{S}_n$ having $|\Gamma(G)| = \xi$ is associated with the $N(G)$-th column of $M(\mathcal{S}_n)$ whose column sum is ξ, and each element of $\mathcal{L}_\xi(\mathcal{S}_n)$ is also associated with a column of $M(\mathcal{S}_n)$ whose column sum is ξ. Accordingly, $lsc(n;\xi)$ is the number of columns of $M(\mathcal{S}_n)$ whose column sum is ξ, and also $|\mathcal{L}_\xi(\mathcal{S}_n)|$ is the number of columns of $M(\mathcal{S}_n)$ whose column sum is ξ. Hence we obtain the equality (26), noting Theorem 10. □

Theorem 12. *@ The equality*

$$s_\xi^{(n)} = \frac{\xi}{n!}|\mathcal{L}_\xi(\mathcal{S}_n)|, \qquad \xi \in \mathcal{D}(\mathcal{S}_n) \tag{27}$$

holds.

Proof. In Theorem 2, let P denote the property that the group-order of the group of a labelled self-complementary graph in $\mathcal{S}_n$ is ξ. Then $\mathcal{K}(P)$ is the set of all labelled self-complementary graphs G in $\mathcal{S}_n$ satisfying $|\Gamma(G)| = \xi$, that is, $|\mathcal{K}(P)| = lsc(n;\xi)$. Hence using Theorem 2, Theorem 10 and Theorem 11, the equality (27) is obtained. □

Let $sc(n)$ be the number of unlabelled self-complementary graphs of order n. A formula on $sc(n)$ has been given by Read[7]. But an enumeration method having been here presented will derive a formula on $sc(n)$, which is stated in the following.

Theorem 13. *The number $sc(n)$ of unlabelled self-complementary graphs of order n is given by*

$$sc(n) = \sum_{\xi \in \mathcal{D}(\mathcal{S}_n)} \frac{\xi}{n!}|\mathcal{L}_\xi(\mathcal{S}_n)|. \tag{28}$$

Proof. Since $sc(n) = \sum_{\xi \in \mathcal{O}(\mathcal{S}_n)} s_\xi^{(n)}$, using Theorem 10 and Theorem 12, we get the equality (28). □

7 Numerical results for self-complementary graphs

This section gives the numerical results for self-complementary graphs of a few small orders n satisfying $n \equiv 0, 1(\mod 4)$. Table 3 gives the numerical examples for labelled self-complementary graphs. Each pair $(\xi,\ lsc(n;\xi))$ in the

Table 3 shows the number $lsc(n;\xi)$ of labelled self-complementary graphs of order n with the group-order ξ. The last column of Table 3 shows the number $lsc(n)$ of labelled self-complementary graphs of order n. On the other hand, Table 4 gives the numerical examples for unlabelled self-complementary graphs. Each pair $(\xi,\, s_\xi^{(n)})$ in the Table 4 shows the number $s_\xi^{(n)}$ of unlabelled self-complementary graphs of order n with the group-order ξ. The last column of Table 4 shows the number $sc(n)$ of unlabelled self-complementary graphs of order n.

Table 3: Labelled self-complementary graphs

n	$(\xi,\, lsc(n;\xi))$	$lsc(n)$
4	$(2, 12)$	12
5	$(2, 60), (10, 12)$	72
8	$(2, 60480), (4, 20160), (8, 15120), (32, 2520)$	98280
9	(2,3265920),(4,544320),(8,226800), (20,36288), (32,45360), (72,5040)	4123728

Table 4: Unlabelled self-complementary graphs

n	$(\xi,\, s_\xi^{(n)})$	$sc(n)$
4	$(2, 1)$	1
5	$(2, 1), (10, 1)$	2
8	$(2, 3), (4, 2), (8, 3), (32, 2)$	10
9	$(2, 18), (4, 6), (8, 5), (20, 2), (32, 4), (72, 1)$	36

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